1 (15 points) Dot and cross product.

a. 2pts \( \vec{A} = (1,1,0), \vec{B} = (1,-1,0) \). Draw these vectors.

b. 11pts Calculate and draw the products \( \vec{A} \cdot \vec{B} \) and \( \vec{A} \times \vec{B} \), showing and explaining all of your work.

c. 2pt Which of these products are scalars and vectors, and how are they related to the original vectors \( \vec{A} = (1,1,0), \vec{B} = (1,-1,0) \)? Describe and discuss.

\[
\vec{A} \cdot \vec{B} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \quad \cos 90^\circ = 2 (0) = 0
\]

\[
\vec{A} \times \vec{B} = \hat{\imath} (1 \cdot 0 - 1 \cdot 1) + \hat{j} (1 \cdot 0 - 1 \cdot 1) + \hat{k} (1 \cdot 1 - 1 \cdot 1) = \hat{k} \cdot 0 \hat{j} - 2 \hat{k} = \langle 0, 0, -2 \rangle
\]

The \( \vec{A} \times \vec{B} \) by the RHR must lie which way the \( \vec{A} \times \vec{B} \) points, since a RH screw turned from \( \vec{A} + \vec{B} \) emerges out of the page. 

\[
\vec{A} \times \vec{B} = \langle 0, 0, -2 \rangle
\]

Alternatively,

\[
\vec{A} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
ax & ay & az \\
bx & by & bz \\
\end{vmatrix} = \hat{i} (aybz - byaz) - \hat{j} (axbz - bxaz) + \hat{k} (axby - aybx)
\]

\[
= \hat{i} (1 \cdot 0 - 1 \cdot 1) - \hat{j} (1 \cdot 0 - 1 \cdot 1) + \hat{k} (1 \cdot 1 - 1 \cdot 1)
\]

\[
= \hat{k} \cdot 0 \hat{j} - 2 \hat{k} = \langle 0, 0, -2 \rangle
\]

\( \vec{A} \times \vec{B} \) is \( \perp \) to both \( \vec{A} \) and \( \vec{B} \) and is a vector.
1. (15 points) Dot and cross product.
   a. 2pts \( \vec{A} = (1,1,0), \vec{B} = (1,-1,0) \). Draw these vectors. \( \checkmark \)
   b. 11pts Calculate and draw the products \( \vec{A} \cdot \vec{B} \) and \( \vec{A} \times \vec{B} \), showing and explaining all of your work.
   c. 2pts Which of these products are scalars and vectors, and how are they related to the original vectors \( \vec{A} = (1,1,0), \vec{B} = (1,-1,0) \)? Describe and discuss.

---

a)

\[ \text{Z-axis is in face of paper.} \]

\[ \vec{A} \cdot \vec{B} = 1 \]

\[ = A_x B_x + A_y B_y + A_z B_z = (1 \cdot 1) + (1 \cdot 1) + (0 \cdot 0) \]

\[ = 1 + 1 + 0 = 2 \]

\[ \text{redundant check} \]

\[ = |\vec{A}| \cdot |\vec{B}| \cos \theta \]

\[ \vec{A} \text{ forms an } \theta \text{ of } +45^\circ \text{ with } \vec{z} \text{-axis} \]

\[ \vec{B} \text{ forms an } \theta \text{ of } -45^\circ \text{ with } x\text{-axis} \]

\[ \text{Causing a } 90^\circ \text{ x. } \cos 90^\circ = 0 \]

\[ \therefore \vec{A} \cdot \vec{B} = 0 \]

\[ \vec{A} \times \vec{B} = \begin{pmatrix} 0,0,2 \end{pmatrix} \]

\[ \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 0i - 0j + (-1)(-1)k \]

\[ = \begin{pmatrix} 0,0,-2 \end{pmatrix} \]

\(- \text{right hand rule of } \vec{A} \text{ into } \vec{B} \text{ would make } z \text{ into -z again} \)

\[ \text{the paper. Hence a negative } z \text{ value.} \]

\[ = |\vec{A}| \cdot |\vec{B}| \sin \theta = \sqrt{2} \cdot \sqrt{2} \sin 90^\circ \]

\[ \therefore 2 \cdot 1 = 2 \text{ - indirection of Right Hand Rule.} \]

\[ \checkmark \]
c) The dot product $\mathbf{A} \cdot \mathbf{B}$ is scalar.

The cross product $\mathbf{A} \times \mathbf{B}$ is a vector.

The dot product is related to the original vectors by giving a number of how much the two vectors combine in the same direction. Since the vectors are 1 to each other, the dot product is zero.

The cross product is related to the original vectors by the product of the magnitudes of the original vectors and the magnitude of the resultant vector. The direction of the resultant is found by the right-hand rule and 1 to both original vectors.
1. (15 points) Dot and cross product.
   a. 2pts \( \vec{A} = (1, 1, 0), \vec{B} = (1, -1, 0) \). Draw these vectors.
   b. 11pts Calculate and draw the products \( \vec{A} \cdot \vec{B} \) and \( \vec{A} \times \vec{B} \), showing and explaining all of your work.
   c. 2pt Which of these products are scalars and vectors, and how are they related to the original vectors \( \vec{A} = (1,1,0), \vec{B} = (1,-1,0) \)? Describe and discuss.

\[ \vec{A} \cdot \vec{B} = <1,1,0> \cdot <1,-1,0> \]
\[ = (1)(1) + (1)(-1) + (0)(0) = 0 \]

The dot product \( \vec{A} \cdot \vec{B} \) comes out to be zero. This is a scalar and it means that the angle between them is 90°, because \( \cos \theta = \frac{\vec{A} \cdot \vec{B}}{||\vec{A}|| \, ||\vec{B}||} \) and \( \cos 90° = 0 \) and \( \frac{0}{||\vec{A}|| \, ||\vec{B}||} = 0 \)

\[ \vec{A} \times \vec{B} \]
\[
\begin{vmatrix}
  i & j & k \\
  1 & 1 & 0 \\
  -1 & 0 & 0 \\
\end{vmatrix}
\]
\[ i [(1)(0) - (0)(-1)] - j [(1)(0) - (0)(1)] + k [(1)(1) - (1)(0)] \]
\[ = i[0] - j[0] + k[1] = 0i - 0j + k \]
\[ \vec{A} \times \vec{B} = <0, 0, 1> \]

This answer is a vector, this means the \( z \) component is the only part of the vectors that is \( 1 \).

\[ \cos \theta = \frac{\vec{A} \times \vec{B}}{||\vec{A}|| \, ||\vec{B}||} \]
2. (20 pts) Electric-field and potential.

a. 12 pts Using a sketched figure, your own words and formal mathematics, define electric field and electric potential. Describe their units and give examples of each. Which are vectors and scalar?

b. 8 pts Using an example from class, including a sketched figure and description, explain the phrase "Electric field is the negative gradient of potential". Include an explanation of the standard mathematical notation used for this relationship.

Electric field is force per unit charge. The units are N/C or V/m. \( \overrightarrow{E} \)

Electric potential is energy per unit charge \( \frac{\text{EPE}}{\text{unit charge or c}} \).

The units are Volts. \( V \) is a scalar having only magnitude but not direction. The electric potential lines do not end or begin on a charge. They are in circles.

Electric field lines are vectors and have direction ending or beginning on a charge.

\[ \overrightarrow{E} = \frac{F}{q_0} \]

\[ V = \frac{E\cdot d}{c} \text{ or Volts or N.m} \]

\[ N \cdot m \text{ or F.d = W = Energy} \]

Electric field is the negative gradient of potential.

This means as electric field increases, electric potential decreases.

\[ \overrightarrow{E} = - \overrightarrow{\nabla} V \rightarrow \text{electric potential} \]

\[ \text{Electric field} \rightarrow \text{mag. scalar gradient} \]

Ex: In class we looked at an overhead of Etna's ski slopes. The contour lines represent electrical potentials having higher and lower altitudes.

The closer the lines, the steeper the slope. The closest gravitational field lines represent the closest electric potential lines. When we release a ski at the top of a steep hill, the ski will travel in the direction of the steepest slope (where the electric potential lines are closest).
\[ \text{EPE} = \text{PE} \]

\[ \frac{1}{2} \text{mv}^2 \]

\[ \text{EPE} = gEd \]

\[ \vec{E} \]

\[ \vec{V} \]

\[ \text{slope} = \text{height} \]

\[ \text{skis have higher PE than M2 on the ground} \]

\[ \text{ski goes from high potential to low potential} \]

\[ \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \]

\[ \text{this is what means} \]
2. (20 pts) Electric field and potential.

a. 12 pts Using a sketched figure, your own words and formal mathematics, define electric field and electric potential. Describe their units and give examples of each. Which are vectors and scalar?

b. 8 pts Using an example from class, including a sketched figure and description, explain the phrase "Electric field is the negative gradient of potential". Include an explanation of the standard mathematical notation used for this relationship.

\[\vec{E} = -\nabla V\]

\[E_x = \frac{\partial V}{\partial x}, \quad E_y = \frac{\partial V}{\partial y}, \quad E_z = \frac{\partial V}{\partial z}\]

This means \(\vec{E}\) is equal to negative of the partial derivative that is with respect to the variable. For the \(x\), \(y\), \(z\) components, you must take a derivative of each component separately. This partial derivative allows you to find \(E_x\) by taking the derivative of only the \(x\) component and so on with the \(y\) and \(z\) components.
3. (15pts). Parallel plates. Between two parallel metal plates 3.0mm apart in a vacuum, a voltmeter measured a potential difference of 200V. An electron is released near one plate with zero initial velocity.
   a. 3pts Sketch the above situation. Label your sketch. Sketch two equipotential lines between the plates.
   b. 3pts Calculate and include \( \vec{E} \) on your sketch.
   c. 3pts An electron is released near the lower voltage plate. Describe the motion of the electron in the field. Indicate its path and calculate and label the force(s) it experiences (ignore gravity).
   d. 4pts The electron eventually strikes the higher voltage plate. Calculate its energy and velocity when it strikes this plate.
   e. 2pts A scientist would say the final energy of the electron is path-independent. Use your sketch and your own words to explain what is meant and why this is so.

\[
\begin{align*}
\vec{E} &= \frac{V}{d} = \frac{200 \text{V}}{0.003 \text{m}} = 666666.67 \text{ V/m} \\
\end{align*}
\]
Energy = \( \frac{1}{2}mv^2 \)

\[ E = \frac{1}{2} \times 9.11 \times 10^{-31} \text{ kg} \times \text{velocity}^2 \]

\[ E = 2.2 \times 10^{-17} \text{ J} \]

Energy & electric potential from the charge is 1200 volts. The potential at every point contributes so multiply by three times will cancel leaving us with Joules energy.

I used 2000 because it finds total distance from 2000 charge.

e.)

It is path independent because the electron could travel up and down equally all over, but the only distance that matters is the distance between the plates. This is so because the only thing that matters is the perpendicular distance between the plates.

\[ \sqrt{\text{distance}} \]

\[ \checkmark \text{ good} \]
4. (30 points) \( \vec{E} \) for ring of charge.

a. 25pts A thin ring of radius \( R \) located at the origin has a charge \(-Q\) uniformly distributed over its surface. Starting only from formulas shown on the formula sheet, calculate the electric field due to the ring, both in magnitude and direction, at a location \(<x, 0, 0>\) on the axis of the ring. Show all steps in your work, and completely discuss your reasoning.

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \]

\[ r = \frac{x}{1} = \frac{\langle R \cos \theta, R \sin \theta, z \rangle}{(R^2 + z^2)^{1/2}} \]

\[ \Delta E = |\Delta \vec{E}| = \frac{1}{4\pi\varepsilon_0} \frac{\Delta Q}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{(R^2 + z^2)^{1/2}} \]

\[ \Delta Q \text{ is } Q \left( \frac{\Delta \vec{E}}{2\pi} \right) \text{ integral over } d\theta \quad \text{from } 0 \text{ to } 2\pi \]

\[ E = \sum \Delta E = \int d\vec{E} = \int_{0}^{2\pi} \frac{1}{4\pi\varepsilon_0} \frac{Q}{(R^2 + z^2)^{1/2}} \frac{z}{(R^2 + z^2)^{1/2}} d\theta \]

\[ E = \int_{0}^{2\pi} \frac{Q z}{4\pi\varepsilon_0 (R^2 + z^2)^{3/2}} \]

\[ E = \frac{1}{4\pi\varepsilon_0} \frac{Q z}{(R^2 + z^2)^{3/2}} \]

Steps:

1. Divide charge dist
2. Write \( \Delta E \), \( \Delta \phi \) appropriately
3. \( \Delta E \) on \( \int d\vec{E} \)
4. Check

b. 5pts Check your results three ways, and describe your checks.

1. \( \vec{E} \) is real vector quantity
2. \( \vec{E} \) is force on charge
3. \( \vec{E} \) is divergence of \( \vec{E} \)
4. \( \vec{E} \) is zero outside the ring.