

Measurement Analysis 1: Measurement Uncertainty and Propagation

You should read Sections 1.6 and 1.7 on pp. 12–16 of the Serway text before this activity. Please note that while attending the MA1 evening lecture is optional, the MA1 assignment is NOT optional and must be turned in before the deadline for your division for credit. The deadline for your division is specified in READ ME FIRST! at the front of this manual.

At the end of this activity, you should:

1. Understand the form of measurements in the laboratory, including measured values and uncertainties.
2. Know how to get uncertainties for measurements made using laboratory instruments.
3. Be able to discriminate between measurements that agree and those that are discrepant.
4. Understand the difference between precision and accuracy.
5. Be able to combine measurements and their uncertainties through addition, subtraction, multiplication, and division.
6. Be able to properly round measurements and treat significant figures.

1 Measurements

1.1 Uncertainty in measurements

In an ideal world, measurements are always perfect: there, wooden boards can be cut to exactly two meters in length and a block of steel can have a mass of exactly three kilograms. However, we live in the real world, and here measurements are *never* perfect. In our world, measuring devices have limitations.

The imperfection inherent in all measurements is called an *uncertainty*. In the Physics 152 laboratory, we will write an uncertainty almost every time we make a measurement. Our notation for measurements and their uncertainties takes the following form:

$$(measured\ value \pm uncertainty) proper\ units$$

where the \pm is read ‘plus or minus.’

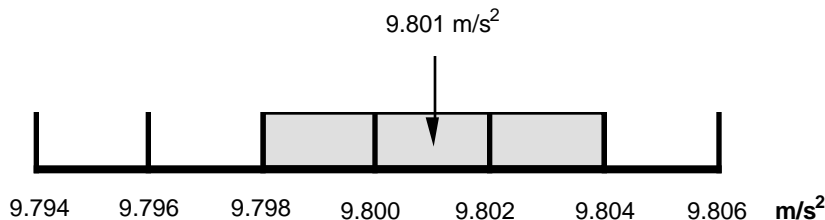


Figure 1: Measurement and uncertainty: $(9.801 \pm 0.003) \text{ m/s}^2$

Consider the measurement $g = (9.801 \pm 0.003) \text{ m/s}^2$. We interpret this measurement as meaning that the experimentally determined value of g can lie anywhere between the values $9.801 + 0.003 \text{ m/s}^2$ and $9.801 - 0.003 \text{ m/s}^2$, or $9.798 \text{ m/s}^2 \leq g \leq 9.804 \text{ m/s}^2$. As you can see, a real world measurement is not one simple measured value, but is actually a *range* of possible values (see Figure 1). This range is determined by the uncertainty in the measurement. As uncertainty is reduced, this range is narrowed.

Here are two examples of measurements:

$$v = (4.000 \pm 0.002) \text{ m/s} \quad G = (6.67 \pm 0.01) \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$$

Look over the measurements given above, paying close attention to the number of decimal places in the measured values and the uncertainties (when the measurement is good to the thousandths place, so is the uncertainty; when the measurement is good to the hundredths place, so is the uncertainty). You should notice that they always agree, and this is most important:

— *In a measurement, the measured value and its uncertainty must always have the same number of digits after the decimal place.*

Examples of nonsensical measurements are $(9.8 \pm 0.0001) \text{ m/s}^2$ and $(9.801 \pm 0.1) \text{ m/s}^2$; writing such nonsensical measurements will cause readers to judge you as either incompetent or sloppy. Avoid writing improper measurements by always making sure the decimal places agree.

Sometimes we want to talk about measurements more generally, and so we write them without actual numbers. In these cases, we use the lowercase Greek letter *delta*, or δ to represent the uncertainty in the measurement. Examples include:

$$(X \pm \delta X) \quad (Y \pm \delta Y)$$

Although units are not explicitly written next to these measurements, they are implied. We will use these general expressions for measurements when we discuss the propagation of uncertainties in Section 4.

1.2 Uncertainties in measurements in lab

In the laboratory you will be taking real world measurements, and for some measurements you will record both measured values and uncertainties. Getting values from measuring

equipment is usually as simple as reading a scale or a digital readout. Determining uncertainties is a bit more challenging since you—not the measuring device—must determine them. When determining an uncertainty from a measuring device, you need to first determine the smallest quantity that can be resolved on the device. Then, for your work in PHYS 152L, the uncertainty in the measurement is taken to be this value. For example, if a digital readout displays 1.35 g, then you should write that measurement as (1.35 ± 0.01) g. The smallest division you can clearly read is your uncertainty.

On the other hand, reading a scale is somewhat subjective. Suppose you use a meter stick that is divided into centimeters to determine the length ($L \pm \delta L$) of a rod, as illustrated in Figure 2. First, you read your measured value from this scale and find that the rod is 6 cm. Depending on the sharpness of your vision, the clarity of the scale, and the boundaries of the measured object, you might read the uncertainty as ± 1 cm, ± 0.5 cm, or ± 0.2 cm. An uncertainty of ± 0.1 cm or smaller is dubious because the ends of the object are rounded and it is hard to resolve ± 0.1 cm. Thus, you might want to record your measurement as $(L \pm \delta L) = (6 \pm 1)$ cm, $(L \pm \delta L) = (6.0 \pm 0.5)$ cm, or $(L \pm \delta L) = (6.0 \pm 0.2)$ cm, since all three measurements would appear reasonable. *For the purposes of discussion and uniformity in this laboratory manual, we will use the largest reasonable uncertainty. For our example, this is ± 1 cm.*

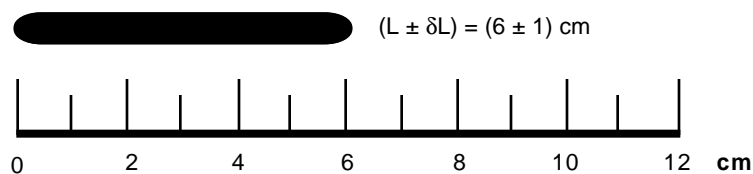


Figure 2: A measurement obtained by reading a scale. Acceptable measurements range from 6.0 ± 0.1 cm to 6.0 ± 0.2 cm, depending on the sharpness of your vision, the clarity of the scale, and the boundaries of the measured object. Examples of unacceptable measurements are 6 ± 2 cm and 6.00 ± 0.01 cm.

1.3 Percentage uncertainty of measurements

When we speak of a measurement, we often want to know how reliable it is. We need some way of judging the relative worth of a measurement, and this is done by finding the percentage uncertainty of a measurement. We will refer to the *percentage uncertainty* of a measurement as the ratio between the measurement's uncertainty and its measured value multiplied by 100%. You will often hear this kind of uncertainty or something closely related used with measurements – a meter is good to $\pm 3\%$ of full scale, or $\pm 1\%$ of the reading, or good to one part in a million.

The *percentage uncertainty* of a measurement ($Z \pm \delta Z$) is defined as $\frac{\delta Z}{Z} \times 100\%$.

Think about percentage uncertainty as a way of telling how much a measurement deviates from “perfection.” With this idea in mind, it makes sense that as the uncertainty

for a measurement decreases, the percentage uncertainty $\frac{\delta Z}{Z} \times 100\%$ decreases, and so the measurement deviates less from perfection. For example, a measurement of (2 ± 1) m has a percentage uncertainty of 50%, or one part in two. In contrast, a measurement of (2.00 ± 0.01) m has a percentage uncertainty of 0.5% (or 1 part in 200) and is therefore the more precise measurement. If there were some way to make this same measurement with zero uncertainty, the percentage uncertainty would equal 0% and there would be no deviation whatsoever from the measured value—we would have a “perfect” measurement. Unfortunately, this never happens in the real world.

1.4 Implied uncertainties

When you read a physics textbook, you may notice that almost all the measurements stated are missing uncertainties. Does this mean that the author is able to measure things perfectly, without any uncertainty? Not at all! In fact, it is common practice in textbooks not to write uncertainties with measurements, even though they are actually there. In such cases, the uncertainties are *implied*. We treat these implied uncertainties the same way as we did when taking measurements in lab:

— *In a measurement with an implied uncertainty, the actual uncertainty is written as ± 1 in the smallest place value of the given measured value.*

For example, if you read $g = 9.80146$ m/s² in a textbook, you know this measured value has an implied uncertainty of 0.00001 m/s². To be more specific, you could then write $(g \pm \delta g) = (9.80146 \pm 0.00001)$ m/s².

1.5 Decimal points — don’t lose them

If a decimal point gets lost, it can have disastrous consequences. One of the most common places where a decimal point gets lost is in front of a number. For example, writing .52 cm sometimes results in a reader missing the decimal point, and reading it as 52 cm — one hundred times larger! After all, a decimal point is only a simple small dot. However, writing 0.52 cm virtually eliminates the problem, and writing leading zeros for decimal numbers is standard scientific and engineering practice.

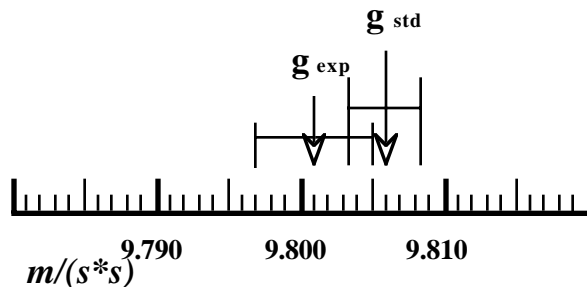
2 Agreement, Discrepancy, and Difference

In the laboratory, you will not only be taking measurements, but also comparing them. You will compare your experimental measurements (i.e. the ones you find in lab) to some theoretical, predicted, or standard measurements (i.e. the type you calculate or look up in a textbook) as well as to experimental measurements you make during a second (or third...) data run. We need a method to determine how closely these measurements compare.

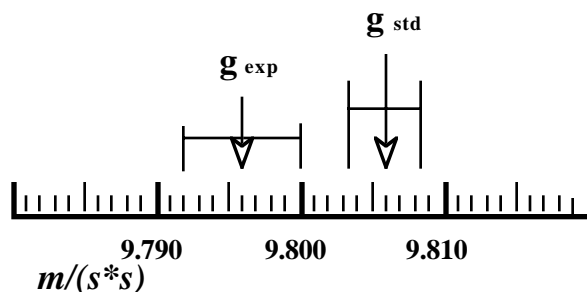
To simplify this process, we adopt the following notion: two measurements, when compared, either *agree* within experimental uncertainty or they are *discrepant* (that is, they do

not agree). Before we illustrate how this classification is carried out, you should first recall that a measurement in the laboratory is not made up of one single value, but a whole range of values. With this in mind, we can say,

Two measurements are in *agreement* if the two measurements share values in common; that is, their respective uncertainty ranges partially (or totally) overlap.



a: two values in experimental agreement



b: two discrepant values

Figure 3: Agreement and discrepancy of gravity measurements

For example, a laboratory measurement of $(g_{exp} \pm \delta g_{exp}) = (9.801 \pm 0.004) \text{ m/s}^2$ is being compared to a scientific standard value of $(g_{std} \pm \delta g_{std}) = (9.8060 \pm 0.0025) \text{ m/s}^2$. As illustrated in Figure 3(a), we see that the ranges of the measurements partially overlap, and so we conclude that the two measurements agree.

Remember that measurements are either in agreement or are discrepant. It then makes sense that,

Two measurements are *discrepant* if the two measurements *do not* share values in common; that is, their respective uncertainty ranges do not overlap.

Suppose as an example that a laboratory measurement $(g_{exp} \pm \delta g_{exp}) = (9.796 \pm 0.004) \text{ m/s}^2$ is being compared to the value of $(g_{std} \pm \delta g_{std}) = (9.8060 \pm 0.0025) \text{ m/s}^2$. From Figure 3 (b) we notice that the ranges of the measurements do not overlap at all, and so we say these measurements are discrepant.

When two measurements being compared do not agree, we want to know by how much they do not agree. We call this quantity the *discrepancy* between measurements, and we use the following formula to compute it:

The discrepancy Z between an experimental measurement ($X \pm \delta X$) and a theoretical or standard measurement ($Y \pm \delta Y$) is:

$$Z = \frac{X_{\text{experimental}} - Y_{\text{standard}}}{Y_{\text{standard}}} \times 100\%$$

As an example, take the two discrepant measurements ($g_{\text{exp}} \pm \delta g_{\text{exp}}$) and ($g_{\text{std}} \pm \delta g_{\text{std}}$) from the previous example. Since we found that these two measurements are discrepant, we can calculate the discrepancy Z between them as:

$$Z = \frac{g_{\text{exp}} - g_{\text{std}}}{g_{\text{std}}} \times 100\% = \frac{9.796 - 9.8060}{9.8060} \times 100\% \approx -0.10\%$$

Keep the following in mind when comparing measurements in the laboratory:

1. If you find that two measurements agree, state this in your report. Do NOT compute a discrepancy.
2. If you find that two measurements are discrepant, state this in your report and then go on to compute the discrepancy.

3 Precision and accuracy

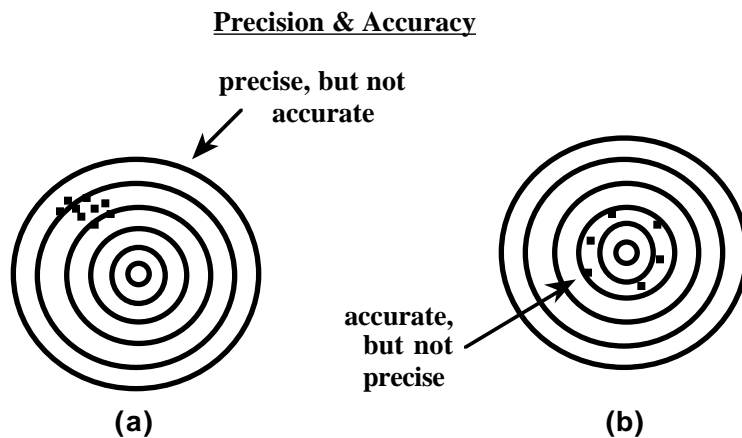


Figure 4: Precision and accuracy in target shooting.

In everyday language, the words precision and accuracy are often interchangeable. In the sciences, however, the two terms have distinct meanings:

Precision describes the degree of certainty one has about a measurement.

Accuracy describes how well measurements agree with a known, standard measurement.

Let's first examine the concept of precision. Figure 4(a) shows a precise target shooter, since all the shots are close to one another. Because all the shots are clustered about a single point, there is a high degree of certainty in where the shots have gone and so therefore the shots are precise. In Figure 5(b), the measurements on the ruler are all close to one another, and like the target shots, they are precise as well.

Accuracy, on the other hand, describes how well something agrees with a standard. In Figure 4(b), the “standard” is the center of the target. All the shots are close to this center, and so we would say that the targetshooter is accurate. However, the shots are not close to one another, and so they are not precise. Here we see that the terms “precision” and “accuracy” are definitely not interchangeable; one does not imply the other. Nevertheless, it is possible for something to be both accurate and precise. In Figure 5(c), the measurements are accurate, since they are all close to the “standard” measurement of 1.5 cm. In addition, the measurements are precise, because they are all clustered about one another.

Note that it is also possible for a measurement to be neither precise nor accurate. In Figure 5(a), the measurements are neither close to one another (and therefore not precise), nor are they close to the accepted value of 1.5 cm (and hence not accurate).

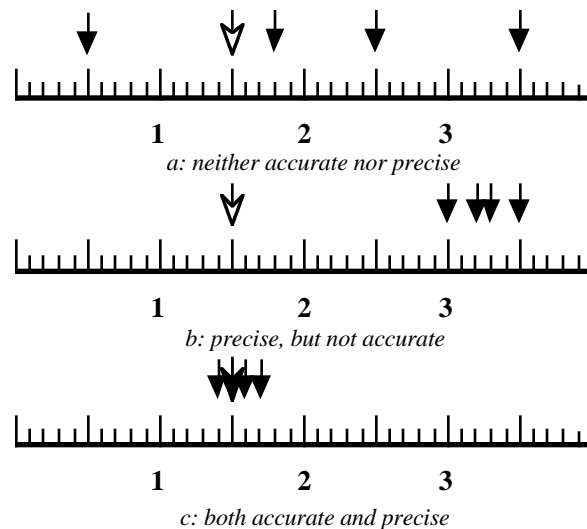


Figure 5: Examples of precision and accuracy in length measurements. Here the hollow headed arrows indicate the ‘actual’ value of 1.5 cm. The solid arrows represent measurements.

You may have noticed that we have already developed techniques to measure precision and accuracy. In Section 1.3, we compared the uncertainty of a measurement to its measured value to find the percentage uncertainty. The calculation of percentage uncertainty is actually a test to determine how certain you are about a measurement; in other words, how precise the measurement is. In Section 2, we learned how to compare a measurement to a standard

or accepted value by calculating a percent discrepancy. This comparison told you how close your measurement was to this standard measurement, and so finding percent discrepancy is really a test for accuracy.

It turns out that in the laboratory, precision is much easier to achieve than accuracy. Precision can be achieved by careful techniques and handiwork, but accuracy requires excellence in experimental design and measurement analysis. During this laboratory course, you will examine both accuracy and precision in your measurements and suggest methods of improving both.

4 Propagation of uncertainty (worst case)

In the laboratory, we will need to combine measurements using addition, subtraction, multiplication, and division. However, measurements are composed of two parts—a measured value and an uncertainty—and so any algebraic combination must account for both. Performing these operations on the measured values is easily accomplished; handling uncertainties poses the challenge. We make use of the *propagation of uncertainty* to combine measurements with the assumption that as measurements are combined, uncertainty increases—hence the uncertainty *propagates* through the calculation. Here we show how to combine two measurements and their uncertainties. Often in lab you will have to keep using the propagation formulae over and over, building up more and more uncertainty as you combine three, four or five set of numbers.

1. **When adding two measurements**, the uncertainty in the final measurement is the sum of the uncertainties in the original measurements:

$$(A \pm \delta A) + (B \pm \delta B) = (A + B) \pm (\delta A + \delta B) \quad (1)$$

As an example, let us calculate the combined length $(L \pm \delta L)$ of two tables whose lengths are $(L_1 \pm \delta L_1) = (3.04 \pm 0.04)$ m and $(L_2 \pm \delta L_2) = (10.30 \pm 0.01)$ m. Using this addition rule, we find that

$$(L \pm \delta L) = (3.04 \pm 0.04) \text{ m} + (10.30 \pm 0.01) \text{ m} = (13.34 \pm 0.05) \text{ m}$$

2. **When subtracting two measurements**, the uncertainty in the final measurement is again equal to the sum of the uncertainties in the original measurements:

$$(A \pm \delta A) - (B \pm \delta B) = (A - B) \pm (\delta A + \delta B) \quad (2)$$

For example, the difference in length between the two tables mentioned above is

$$\begin{aligned} (L_2 \pm \delta L_2) - (L_1 \pm \delta L_1) &= (10.30 \pm 0.01) \text{ m} - (3.04 \pm 0.04) \text{ m} \\ &= [(10.30 - 3.04) \pm (0.01 + 0.04)] \text{ m} \\ &= (7.26 \pm 0.05) \text{ m} \end{aligned}$$

Be careful not to subtract uncertainties when subtracting measurements— uncertainty ALWAYS gets worse as more measurements are combined.

3. **When multiplying two measurements**, the uncertainty in the final measurement is found by summing the percentage uncertainties of the original measurements and then multiplying that sum by the product of the measured values:

$$(A \pm \delta A) \times (B \pm \delta B) = (AB) \left[1 \pm \left(\frac{\delta A}{A} + \frac{\delta B}{B} \right) \right] \quad (3)$$

A quick derivation of this multiplication rule is given below. First, assume that the measured values are large compared to the uncertainties; that is, $A \gg \delta A$ and $B \gg \delta B$. Then, using the distributive law of multiplication:

$$\begin{aligned} (A \pm \delta A) \times (B \pm \delta B) &= AB + A(\pm\delta B) + B(\pm\delta A) + (\pm\delta A)(\pm\delta B) \\ &\cong AB \pm (A \delta B + B \delta A) \end{aligned} \quad (4)$$

Since the uncertainties are small compared to the measured values, the product of two small uncertainties is an even smaller number, and so we discard the product $(\pm\delta A)(\pm\delta B)$. With further simplification, we find:

$$\begin{aligned} AB + A(\pm\delta B) + B(\pm\delta A) &= AB + B(\pm\delta A) + A(\pm\delta B) \\ &= AB \left[1 \pm \left(\frac{\delta A}{A} + \frac{\delta B}{B} \right) \right] \end{aligned}$$

It should be noted that the above equation is mathematically undefined if either A or B is zero. In this case equation 4 is used to obtain the uncertainty since it is valid for all values of A and B.

Now let us use the multiplication rule to determine the area of a rectangular sheet with length $(l \pm \delta l) = (1.50 \pm 0.02)$ m and width $(w \pm \delta w) = (20 \pm 1)$ cm = (0.20 ± 0.01) m. The area $(A \pm \delta A)$ is then

$$\begin{aligned} (A \pm \delta A) &= (l \pm \delta l) \times (w \pm \delta w) = (lw) \left[1 \pm \left(\frac{\delta l}{l} + \frac{\delta w}{w} \right) \right] \\ &= (1.50 \times 0.20) \left[1 \pm \left(\frac{0.02}{1.50} + \frac{0.01}{0.20} \right) \right] \text{ m}^2 \\ &= 0.300[1 \pm (0.0133 + 0.0500)] \text{ m}^2 = 0.300[1 \pm 0.0633] \text{ m}^2 \\ &= (0.300 \pm 0.0190) \text{ m}^2 \\ &\approx (0.30 \pm 0.02) \text{ m}^2 \end{aligned}$$

Notice that the final values for uncertainty in the above calculation were determined by multiplying the product (lw) outside the bracket by the sum of the two percentage uncertainties ($\delta l/l + \delta w/w$) inside the bracket. Always remember this crucial step! Also, notice how the final measurement for the area was rounded. This rounding was performed by following the rules of significant figures, which are explained in detail later in Section 5.

Recall our discussion of percentage uncertainty in Section 1.3. It is here that we see the benefits of using such a quantity; specifically, we can use it to tell right away which of the two original measurements contributed most to the final area uncertainty. In the above example, we see that the percentage uncertainty of the width measurement ($\delta w/w \times 100\%$) is 5%, which is larger than the percentage uncertainty ($\delta l/l \times 100\% \approx 1.3\%$) of the length measurement. Hence, the width measurement contributed most to the final area uncertainty, and so if we wanted to improve the precision of our area measurement, we should concentrate on reducing width uncertainty δw (since it would have a greater effect on the total uncertainty) by changing our method for measuring width.

4. **When dividing two measurements**, the uncertainty in the final measurement is found by summing the percentage uncertainties of the original measurements and then multiplying that sum by the quotient of the measured values:

$$\frac{(A \pm \delta A)}{(B \pm \delta B)} = \left(\frac{A}{B}\right) \left[1 \pm \left(\frac{\delta A}{A} + \frac{\delta B}{B}\right)\right] \quad (5)$$

As an example, let's calculate the average speed of a runner who travels a distance of (100.0 ± 0.2) m in (9.85 ± 0.12) s using the equation $\bar{v} = D/t$, where \bar{v} is the average speed, D is the distance traveled, and t is the time it takes to travel that distance.

$$\begin{aligned} \bar{v} &= \frac{D \pm \delta D}{t \pm \delta t} = \left(\frac{D}{t}\right) \left[1 \pm \left(\frac{\delta D}{D} + \frac{\delta t}{t}\right)\right] \\ &= \left(\frac{100.0 \text{ m}}{9.85 \text{ s}}\right) \left[1 \pm \left(\frac{0.2}{100.0} + \frac{0.12}{9.85}\right)\right] \\ &= 10.15 [1 \pm (0.002000 + 0.01218)] \text{ m/s} = 10.15 [1 \pm (0.01418)] \text{ m/s} \\ &= (10.15 \pm 0.1439) \text{ m/s} \\ &\approx (10.2 \pm 0.1) \text{ m/s} \end{aligned}$$

In this particular example the final uncertainty results mainly from the uncertainty in the measurement of t , which is seen by comparing the percentage uncertainties of the time and distance measurements, $(\delta t/t) \approx 1.22\%$ and $(\delta D/D) \approx 0.20\%$, respectively. Therefore, to reduce the uncertainty in $(\bar{v} \pm \delta \bar{v})$, we would want to look first at changing the way t is measured.

5. **Special cases—inversion and multiplication by a constant:**

- (a) If you have a quantity $X \pm \delta X$, you can invert it and apply the original percentage uncertainty:

$$\frac{1}{X \pm \delta X} = \left(\frac{1}{X}\right) \left[1 \pm \frac{\delta X}{X}\right]$$

- (b) To multiply by a constant,

$$k \times (Y \pm \delta Y) = [kY \pm k\delta Y]$$

It is important to realize that these formulas and techniques allow you to perform the four basic arithmetic operations. You can (and will) combine them by repetition for the sum of three measurements, or the cube of a measurement. Normally it is impossible to use these simple rules for more complicated operations such as a square root or a logarithm, but the trigonometric functions $\sin \theta$, $\cos \theta$, and $\tan \theta$ are exceptions. Because these functions are defined as the ratios between lengths, we can use the quotient rule to evaluate them. For example, in a right triangle with opposite side $(x \pm \delta x)$ and hypotenuse $(h \pm \delta h)$, $\sin \theta = \frac{(x \pm \delta x)}{(h \pm \delta h)}$. Similarly, any expression that can be broken down into arithmetic steps may be evaluated with these formulas; for example, $(x \pm \delta x)^2 = (x \pm \delta x)(x \pm \delta x)$.

6. Finding the uncertainty of a square root

The method for obtaining the square root of a measurement. uses some algebra coupled with the multiplication rule.

Let $(A \pm \delta A)$ and $(B \pm \delta B)$ be two measurements. Further, assume that the square root of $(A \pm \delta A)$ is equal to the measurement $(B \pm \delta B)$. Then,

$$\sqrt{(A \pm \delta A)} = (B \pm \delta B) \tag{6}$$

Squaring both sides, we obtain

$$(A \pm \delta A) = (B \pm \delta B)^2$$

Using the multiplication rule on $(B \pm \delta B)^2$, we find

$$\begin{aligned} (A \pm \delta A) &= (B \pm \delta B)^2 \\ &= B^2 \left[1 \pm \left(\frac{2\delta B}{B}\right)\right] \\ &= (B^2 \pm 2B\delta B) \end{aligned}$$

Thus,

$$(A \pm \delta A) = (B^2 \pm 2B\delta B) \text{ which means } B = \sqrt{A} \text{ and } \delta B = \frac{\delta A}{2B} = \frac{\delta A}{2\sqrt{A}}.$$

Making this substitution into Equation 6, we arrive at the final result

$$\sqrt{(A \pm \delta A)} = \left(\sqrt{A} \pm \frac{\delta A}{2\sqrt{A}} \right) \quad (7)$$

This technique for finding the uncertainty in a square root will be required in E4 — E6.

7. **Another example: a case involving a triple product.** The formula for the volume of a rod with a circular cross-section (πr^2) and length l is given by $V = \pi r^2 l$. Given initial measurements ($r \pm \delta r$) and ($l \pm \delta l$), derive an expression for ($V \pm \delta V$). Note that π has *no* uncertainty.

Using the derivation of the worst case multiplication propagation rule (Equation 4) as a guide, we start with

$$(V \pm \delta V) = \pi(r \pm \delta r)(r \pm \delta r)(l \pm \delta l)$$

and expand the terms involving r on the left hand side.

$$(V \pm \delta V) = \pi[r^2 + r(\pm\delta r) + r(\pm\delta r) + (\pm\delta r)(\pm\delta r)](l \pm \delta l)$$

Discarding the term involving the product of measurement uncertainties (δr)(δr), since it is small compared to the other terms, we obtain

$$(V \pm \delta V) = \pi[r^2 + 2r(\pm\delta r)](l \pm \delta l)$$

Next we multiply out the final product on the left.

$$(V \pm \delta V) = \pi[r^2 l + r^2(\pm\delta l) + 2rl(\pm\delta r) + 2r(\pm\delta r)(\pm\delta l)]$$

Again we discard terms involving products of measurement uncertainties such as (δr)(δl) to obtain

$$(V \pm \delta V) = \pi[r^2 l + r^2(\pm\delta l) + 2rl(\pm\delta r)]$$

Finally, we can factor out $r^2 l$ to obtain

$$(V \pm \delta V) = \pi r^2 l \left(1 \pm \frac{\delta l}{l} \pm 2 \frac{\delta r}{r} \right)$$

8. **Other uncertainty propagation techniques.** The worst case uncertainty propagation assumes that all measurement uncertainties conspire to give the worst possible uncertainty in your final result. Fortunately this does not usually happen in nature, and there are techniques to take this into account, the simplest being the addition of uncertainties in quadrature and taking the square root of the sum. However, these techniques are more complex and inconsistent with the mathematical requirements for PHYS 152, and we have avoided them. A good start in learning about these more sophisticated techniques is to read the references listed at the end of this chapter.

5 Rounding measurements

The previous sections contain the bulk of what you need to take and analyze measurements in the laboratory. Now it is time to discuss the finer details of measurement analysis. The subtleties we are about to present cause an inordinate amount of confusion in the laboratory. Getting caught up in details is a frustrating experience, and the following guidelines should help alleviate these problems.

An often-asked question is, “How should I round my measurements in the laboratory?” **The answer is that you must watch significant figures in calculations and then be sure the number of decimal places of a measured value and its uncertainty agree.** Before we give an example, we should explore these two ideas in some detail.

5.1 Treating significant figures

The simplest definition for a *significant figure* is a digit (0 - 9) that actually represents some quantity. Zeros that are used to locate a decimal point are not considered significant figures. Any measured value, then, has a specific number of significant figures. See Table 1 for examples.

There are two major rules for handling significant figures in calculations. One applies for addition and subtraction, the other for multiplication and division.

1. **When adding or subtracting quantities**, the number of decimal places in the result should equal the smallest number of decimal places of any term in the sum (or difference).

Examples:

$$51.4 - 1.67 = 49.7 \quad 7146 - 12.8 = 7133 \quad 20.8 + 18.72 + 0.851 = 40.4$$

2. **When multiplying or dividing quantities**, the number of significant figures in the final answer is the same as the number of significant figures in the least accurate of the quantities being multiplied (or divided).

Examples:

$$2.6 \times 31.7 = 82 \text{ not } 82.42 \quad 5.3 \div 748 = 0.0071 \text{ not } 0.007085561$$

5.2 Measured values and uncertainties: Number of decimal places

As mentioned earlier in Section 1.1, we learned that for any measurement ($X \pm \delta X$), the number of decimal places of the measured value X must equal those of the corresponding uncertainty δX .

Below are some examples of correctly written measurements. Notice how the number of decimal places of the measured value and its corresponding uncertainty agree.

$$(L \pm \delta L) = (3.004 \pm 0.002) \text{ m} \quad (m \pm \delta m) = (41.2 \pm 0.4) \text{ kg}$$

Measured value	Number of significant figures
123	3
1.23	3
1.230	4
0.00123	3
0.001230	4

Table 1: Examples of significant figures

5.3 Rounding

Suppose we are asked to find the area ($A \pm \delta A$) of a rectangle with length ($l \pm \delta l$) = (2.708 \pm 0.005) m and width ($w \pm \delta w$) = (1.05 \pm 0.01) m. Before propagating the uncertainties by using the multiplication rule, we should first figure out how many significant figures our final measured value A must have. In this case, $A = lw$, and since l has four significant figures and w has three significant figures, A is limited to three significant figures. Remember this result; we will come back to it in a few steps.

We may now use the multiplication rule to calculate the area:

$$\begin{aligned}
 (A \pm \delta A) &= (l \pm \delta l) \times (w \pm \delta w) \\
 &= (lw) \left[1 \pm \left(\frac{\delta l}{l} + \frac{\delta w}{w} \right) \right] \\
 &= (2.708 \times 1.05) \left[1 \pm \left(\frac{0.005}{2.708} + \frac{0.01}{1.05} \right) \right] \text{ m}^2 \\
 &= (2.843) [1 \pm (0.001846 + 0.009524)] \text{ m}^2 \\
 &= 2.843 (1 \pm 0.011370) \text{ m}^2 \\
 &= (2.843 \pm 0.03232) \text{ m}^2
 \end{aligned}$$

Notice that in the intermediate step directly above, we allowed each number one extra significant figure beyond what we know our final measured value will have; that is, we know the final value will have three significant figures, but we have written each of these intermediate numbers with four significant figures. Carrying the extra significant figure ensures that we will not introduce round-off error.

We are just two steps away from writing our final measurement. Step one is recalling the result we found earlier—that our final measured value must have three significant figures. Thus, we will round 2.843 m² to 2.84 m². Once this step is accomplished, we round our uncertainty to match the number of decimal places in the measured value. In this case, we round 0.03233 m² to 0.03 m². Finally, we can write

$$(A \pm \delta A) = (2.84 \pm 0.03) \text{ m}^2$$

6 References

These books are on reserve in the Physics Library (PHYS 291). Ask for them by the author's last name.

1. Bevington, P. R., *Data reduction and uncertainty analysis for the physical sciences* (McGraw-Hill, New York, 1969).
2. Young, H. D., *Statistical treatment of experimental data* (McGraw-Hill, New York, 1962).

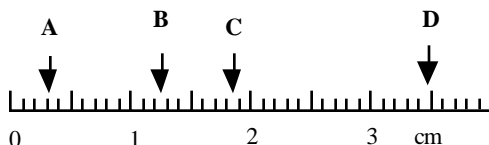
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Measurement Analysis Problem Set MA1

Name _____ Lab day/time _____

Division _____ GTA _____

Write your answers in the space provided. *For full credit, show all essential intermediate steps, include units and ensure that measured values and uncertainties agree in the number of decimal places.*



1. Using the above figure, write the measured values and uncertainties for the following:

(a) $A = (\quad \pm \quad) \text{ cm.}$

(b) $B = (\quad \pm \quad) \text{ cm.}$

(c) $C = (\quad \pm \quad) \text{ cm.}$

(d) $D = (\quad \pm \quad) \text{ cm.}$

(e) How did you determine these uncertainties?

2. In the laboratory, your partner uses a digital balance to find the mass of a small object. He/she tells you (correctly) that the digital readout shows 43 g.

(a) Write the correct mass measurement and its uncertainty.

$(M \pm \delta M) = (\quad \pm \quad) \text{ g.}$

(b) What is the percentage uncertainty of this measurement?

3. In lab, one of your partners determines (correctly) that the surface area of an object is 25.97 cm^2 . All of the figures in this measurement are significant.

(a) Your HP-9000 calculator tells you that the uncertainty is 0.04382361 cm^2 . Write the appropriate measurement and its uncertainty.

$$(A \pm \delta A) = (\quad \pm \quad) \text{ cm}^2$$

(b) Alternatively, your HP-9000 told you that the uncertainty is $0.0012543797 \text{ cm}^2$. Write the appropriate measurement and its uncertainty.

$$(A \pm \delta A) = (\quad \pm \quad) \text{ cm}^2$$

(c) Finally, the calculator told you that the uncertainty is 0.017386642 cm^2 . Write the appropriate measurement and its uncertainty.

$$(A \pm \delta A) = (\quad \pm \quad) \text{ cm}^2$$

(d) Briefly explain how you determined these three numerical uncertainties.

4. In the laboratory you determine the gravitational constant ($g_{exp} \pm \delta g_{exp}$) to be $(9.824 \pm 0.006) \text{ m/s}^2$. According to a geophysical survey, the accepted local value for ($g_{acc} \pm \delta g_{acc}$) is $(9.802 \pm 0.002) \text{ m/s}^2$.

(a) Draw a diagram like Figure 3 showing whether your measurement agrees with the accepted value within the limits of experimental uncertainty or not.

(b) If these measures do not agree, what is the actual discrepancy?

5. For a laboratory exercise, you determine the masses of two airtrack gliders as $(m_1 \pm \delta m_1) = (484.9 \pm 0.3)$ g and $(m_2 \pm \delta m_2) = (314.2 \pm 0.1)$ g. Determine the following combinations of the measurements, and show your work.

(a) $(m_2 \pm \delta m_2) - (m_1 \pm \delta m_1) = (\quad \pm \quad)$ g.

- (b) Determine the precisions of the measurements $(m_1 \pm \delta m_1)$ and $(m_2 \pm \delta m_2)$. Which calculated precision is the larger? Using this information, determine which is the better measurement.

(c) $(m_1 \pm \delta m_1) + (m_2 \pm \delta m_2) = (\quad \pm \quad)$ g.

6. (a) Laboratory measurements performed upon a rectangular steel plate show the length $(l \pm \delta l) = (2.41 \pm 0.25)$ cm and the width $(w \pm \delta w) = (8.30 \pm 0.10)$ cm. Determine the area of the steel plate and the uncertainty in the area.

$A \pm \delta A = (\quad \pm \quad)$ cm².

- (b) As Director of the National Science Foundation, you must decide what is the best means to improve the precision of the area measurement of the steel plate. You can spend money on a space gizmotron to better measure length, or on a superconducting whizbang to better measure width, but not both. On which measurement should you spend the money? Justify your decision with numbers.

7. During another airtrack experiment, you determine the initial position of a glider to be $(x_i \pm \delta x_i) = (0.482 \pm 0.001)$ m and its final position to be $(x_f \pm \delta x_f) = (0.633 \pm 0.003)$ m, with an elapsed time $(t \pm \delta t) = (0.48 \pm 0.01)$ s during the displacement.

(a) Find the total displacement of the glider $(D \pm \delta D) = (x_f \pm \delta x_f) - (x_i \pm \delta x_i)$.
 $(D \pm \delta D) = (\quad \pm \quad)$ m.

(b) Find the average speed of the glider $(\bar{v}_{glider} \pm \delta \bar{v}_{glider}) = (D \pm \delta D)/(t \pm \delta t)$.
 $\bar{v}_{glider} \pm \delta \bar{v}_{glider} = (\quad \pm \quad)$ m/s.

- (c) Calculate the percentage uncertainties for the two quantities $(D \pm \delta D)$ and $(t \pm \delta t)$. Based on these precisions, determine which of the two quantities contributes most to the overall uncertainty $\delta \bar{v}_{glider}$.

- (d) If we could change the apparatus so as to measure *either* distance *or* time ten times more accurately (but not both), which should we change and why?

8. In the laboratory, a measurement for $(x \pm \delta x)$ was taken as $(x \pm \delta x) = (33.4 \pm 0.5) \text{ s}^2$. Write a value for its reciprocal. Show calculations, and ensure that you handle significant figures properly.

$$\frac{1}{(x \pm \delta x)} = (\quad \pm \quad) \frac{1}{\text{s}^2}.$$

9. The formula for the volume of a box with height h , base b , and length l is given by $V = bhl$. Given initial measurements $(h \pm \delta h)$, $(b \pm \delta b)$, and $(l \pm \delta l)$, derive an expression for $(V \pm \delta V)$. Do NOT use the multiplication rule (Equation 3) in deriving this equation.

Hint: Use the *derivation* of the worst case multiplication propagation rule (Equation 4) as a guide. Start with $(V \pm \delta V) = (b \pm \delta b)(h \pm \delta h)(l \pm \delta l)$. Expand, show intermediate steps, and regroup and simplify your solution as much as possible. Discard products of measurement uncertainties, such as $(\delta h)(\delta b)(l)$ and $(\delta h)(\delta b)(\delta l)$.