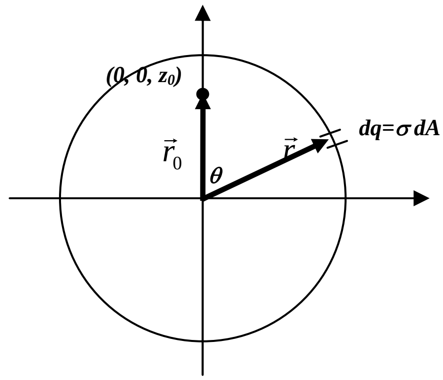
One of the first examples texts use to illustrate Gauss' law is to show that the field due to a thin spherical shell of charge is zero everywhere inside the shell and equivalent to the field from a point charge everywhere outside. Authors also qualitatively argue that the field should be zero, but the result is rarely shown with direct integration. This document shows how to directly integrate Coulomb's law to get the result described above.



The diagram at right shows the problem. In general, the contribution to the field at the position represented by $\vec{r}_0$ due to a small amount of charge dq at position $\vec{r}$ is given by

$$d\vec{E} = k\frac{dq}{|\vec{r}_0 - \vec{r}|^3}(\vec{r}_0 - \vec{r})$$

The field at some point on the z-axis due to any small section of the shell becomes

$$d\vec{E} = k\frac{dq}{|\vec{r}_0 - \vec{r}|^3}(\vec{r}_0 - \vec{r}) = k\frac{dq}{|\vec{r}_0 - \vec{r}|^3}[(0-x)\hat{x} + (0-y)\hat{y} + (z_0 - z)\hat{z}]$$

Given the symmetry it seems reasonable to assume that the x- and y-components of the field are zero (i.e. $\vec{E} = E_z\hat{z}$), so I'll focus on the z-component of $d\vec{E}$:

$$dE_z = k\frac{dq}{|\vec{r}_0 - \vec{r}|^3}(z_0 - z)$$

To get the total field to the entire shell, we'll need to integrate and spherical coordinates are likely to be easier than other choices. To do the integral, we'll need:

- Definition of charge density: $\sigma \equiv q/A$ (and consequentially $dq = \sigma dA$)
- Some spherical coordinates details:
  - $dA = R^2 d\phi \sin\theta d\theta$ and $A = 4\pi R^2$
  - limits of integration: $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$
  - $z = R\cos\theta$ on the surface of the sphere
- Law of cosines to express $|\vec{r}_0 - \vec{r}|$ for our system: $|\vec{r}_0 - \vec{r}| = (z_0^2 + R^2 - 2z_0 R\cos\theta)^{1/2}$

Combining these elements:

$$E_z = \int_{shell} k\frac{dq}{|\vec{r}_0 - \vec{r}|^3}(z_0 - z) = k\int_0^{2\pi} d\phi \int_0^{\pi} \sigma R^2(\sin\theta)d\theta \frac{z_0 - R\cos\theta}{(z_0^2 + R^2 - 2z_0 R\cos\theta)^{3/2}}$$

Use some substitutions ($\varepsilon \equiv z_0/R$ and $u \equiv -\cos\theta,\ du = [\sin\theta]d\theta$) to simplify the appearance of the integral:

$$E_z = k\sigma\int_0^{2\pi} d\phi \int_{-\cos 0}^{-\cos\pi} du \frac{\varepsilon + u}{[1 + \varepsilon^2 + 2\varepsilon u]^{3/2}}$$

Do the integration on ϕ to get:

$$E_z = 2\pi k\sigma \int_{-1}^{1} du \frac{\varepsilon + u}{[1+\varepsilon^2 + 2\varepsilon u]^{3/2}} = 2\pi k\sigma \left[\int_{-1}^{1} du \frac{\varepsilon}{[1+\varepsilon^2 + 2\varepsilon u]^{3/2}} + \int_{-1}^{1} du \frac{u}{[1+\varepsilon^2 + 2\varepsilon u]^{3/2}} \right]$$

The first term inside the brackets also integrates pretty easily:

$$I_1 \equiv \varepsilon \int_{-1}^{1} du [1+\varepsilon^2 + 2\varepsilon u]^{-3/2} = \varepsilon [1+\varepsilon^2 + 2\varepsilon u]^{-1/2} (-2) \frac{1}{2\varepsilon}\Bigg|_{-1}^{+1} = -[1+\varepsilon^2 + 2\varepsilon u]^{-1/2}\Big|_{-1}^{+1}$$

The second term requires integration by parts ($\int u dv = uv - \int v du$). If you choose $dv = [1+\varepsilon^2 + 2\varepsilon u]^{-3/2} du$ and $u = u$, then the integral needed to find v is essentially I_1:

$$v = \int du [1+\varepsilon^2 + 2\varepsilon u]^{-1/2} = \frac{I_1}{\varepsilon} = -\frac{1}{\varepsilon}[1+\varepsilon^2 + 2\varepsilon u]^{-1/2}$$

And then integration by parts gives

$$I_2 \equiv \int_{-1}^{1} u [1+\varepsilon^2 + 2\varepsilon u]^{-3/2} du = -\frac{u}{\varepsilon}[1+\varepsilon^2 + 2\varepsilon u]^{-1/2}\Bigg|_{-1}^{+1} + \frac{1}{\varepsilon}\int_{-1}^{+1} [1+\varepsilon^2 + 2\varepsilon u]^{-1/2} du$$

$$I_2 = -\frac{u}{\varepsilon}[1+\varepsilon^2 + 2\varepsilon u]^{-1/2}\Bigg|_{-1}^{+1} + \frac{1}{\varepsilon^2}[1+\varepsilon^2 + 2\varepsilon u]^{1/2}\Big|_{-1}^{+1}$$

Combining everything:

$$E_z = 2\pi k\sigma \left[-[1+\varepsilon^2 + 2\varepsilon u]^{-1/2} - \frac{u}{\varepsilon}[1+\varepsilon^2 + 2\varepsilon u]^{-1/2} + \frac{1}{\varepsilon^2}[1+\varepsilon^2 + 2\varepsilon u]^{1/2} \right]\Bigg|_{-1}^{+1}$$

$$E_z = 2\pi k\sigma \left[[1+\varepsilon^2 + 2\varepsilon u]^{-1/2} \left(-1 - \frac{u}{\varepsilon} + \frac{1}{\varepsilon^2}[1+\varepsilon^2 + 2\varepsilon u] \right) \right]\Bigg|_{-1}^{+1}$$

$$E_z = 2\pi k\sigma \left[[1+\varepsilon^2 + 2\varepsilon u]^{-1/2} \left(\frac{u}{\varepsilon} + \frac{1}{\varepsilon^2} \right) \right]\Bigg|_{-1}^{+1} = \frac{2\pi k\sigma}{\varepsilon^2} \frac{\varepsilon u + 1}{[1+\varepsilon^2 + 2\varepsilon u]^{1/2}}\Bigg|_{-1}^{+1}$$

Evaluating the result (and being careful about the signs of the square root) gives

$$E_z = \frac{2\pi k\sigma}{\varepsilon^2}\left[\frac{\varepsilon + 1}{|\varepsilon + 1|} - \frac{-\varepsilon + 1}{|\varepsilon - 1|} \right] = \frac{2\pi k\sigma}{\varepsilon^2}\left[\frac{\varepsilon + 1}{|\varepsilon + 1|} + \frac{\varepsilon - 1}{|\varepsilon - 1|} \right]$$

The last step is to show that this answer reduces to what we expect. Before we do, it might be helpful to notice that each term inside the brackets is actually quite simple- any number (except zero) divided by its absolute value is either one (if the number is positive) or negative one (if the number is negative).

Inside the shell

If we are interested in the field at points inside the shell, $|z_0| > R$, and, since $\varepsilon \equiv z_0 / R$, this also means that $|\varepsilon| < 1$. Under these conditions, $(\varepsilon + 1)$ must be positive and $(\varepsilon - 1)$ must be negative, so

$$E_z = \frac{2\pi k\sigma}{\varepsilon^2}\left[\frac{\varepsilon+1}{|\varepsilon+1|} + \frac{\varepsilon-1}{|\varepsilon-1|}\right] = \frac{2\pi k\sigma}{\varepsilon^2}\left[1 + -1\right] = 0$$

Outside the shell

Case 1: At points outside the shell on the positive z-axis, $z_0 > R$ and $\varepsilon > 1$, and both $(\varepsilon + 1)$ and $(\varepsilon - 1)$ are positive so

$$E_z = \frac{2\pi k\sigma}{\varepsilon^2}\left[\frac{\varepsilon+1}{|\varepsilon+1|} + \frac{\varepsilon-1}{|\varepsilon-1|}\right] = \frac{2\pi k\sigma}{\varepsilon^2}\left[1 + 1\right] = \frac{4\pi k\sigma}{\varepsilon^2}$$

Substituting $\varepsilon \equiv z_0 / R$ and $A = 4\pi R^2$ back in, this becomes

$$E_z = \frac{4\pi k\sigma}{\varepsilon^2} = \frac{4\pi k}{\varepsilon^2}\frac{q}{4\pi R^2}\frac{1}{(z_0 / R)^2} = \frac{kq}{z_0^2}$$

Case 2: At points outside the shell on the negative z-axis, both $(\varepsilon + 1)$ and $(\varepsilon - 1)$ are negative and

$$E_z = \frac{2\pi k\sigma}{\varepsilon^2}\left[\frac{\varepsilon+1}{|\varepsilon+1|} + \frac{\varepsilon-1}{|\varepsilon-1|}\right] = \frac{2\pi k\sigma}{\varepsilon^2}\left[-1 + -1\right] = -\frac{4\pi k\sigma}{\varepsilon^2} = -\frac{kq}{z_0^2}$$

The bottom line is that Case 1 and Case 2 have the same interpretation: outside the shell, the electric field points away from the shell with the strength given by Coulomb's law:

$$|E_z| = \frac{kq}{z_0^2}$$

If the reader accepts the original claim from symmetry that the field points only in the z-direction ($\vec{E} = E_z \hat{z}$), we have proven what we set out to prove: that the field inside the shell is zero and the field outside the shell is identical to the field due a point charge at the origin.