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PHYSICS-BASED CALCULUS LESSON

Developing the concept of the limit through calculating average velocity and acceleration of an object in motion.

by

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Physics Department

Abstract

This thesis project aims toward researching and developing an interactive hands-on, small group activity analyzing the motion of an accelerating object. In this lesson, students are led to calculate average and instantaneous velocity and acceleration of an object in free-fall through graphical analysis and the limiting process of calculus. Emphasis is placed on the use of the derivative and average slope function. The lesson's first part makes use of little technology, but it is followed by the second part with the use of motion sensors. Students use curve-fitting to derive functions for the three components of motion and derive part 1 of the Fundamental Theorem of Calculus.

INTRODUCTION

The National Research Council (NRC) and the National Council for Teachers of Mathematics (NCTM) each identify **inquiry** as an effective pedagogy for developing mathematics and science understanding. The NCTM and NRC standards clearly state that students should be engaged in activities demonstrating the use of math and science the promote understanding in these subjects. (Richardson & Liang, 2007, p.3). Since 1991, the NCTM's Professional Standards for Teaching Mathematics has advocated for instruction that is inquiry-based and student-centered (Richardson & Liang, 2007, p. 2).

Over the past 25 years, new approaches to teaching by inquiry developed in physics education have seen progress in advancing the conceptual comprehension of students in introductory physics (McDermott & Redish, 1998). Through this new method of delivery, teachers engage students in interactive, hands-on activities involving concrete observation and analysis in order to anchor abstract concepts to concrete observation in a familiar physical context (Arons, 1997). Through hands-on investigation and discovery, students are more likely to dispel common misconceptions by experiencing phenomena and making connections for themselves.

At the same time, findings in mathematics education suggest that interactive engagement can be a powerful method to help anchor the abstract concepts of math. Traditionally, calculus and physics are taught as separate courses, sparsely utilizing the relationship between the two subjects. Traditional physics courses rely greatly on students' ability to conceptualize natural laws of motion through their understanding of mathematical functions, while many students taking physics courses have not reached a point of mathematical proficiency to solve the sometimes most basic computations, let alone gain insight from the mathematical behavior of a function (Arons, 1997; Stroup, 2005). Newer approaches are having success in improving student's conceptual understanding of physics by removing the math from these activities to allow students to focus on physical observations and form accurate conceptual beliefs about the behavior of natural phenomena. Yet the interconnection between physics and calculus cannot be overemphasized and provides a tremendous opportunity for cross-curriculum instruction. In the past decade, universities have been developing courses that link the physics and mathematics concepts through a common instructional technique (McDermott, Rosenquist & van Zee, 1983).

For example student activities have been designed to demonstrate basics principles of physics through direct observation by which they make conclusions properly aligned with the language of physics (Arons, 1997; McDermott, Rosenquist & van Zee, 1983).

Dr. Allen Emerson of St John Fisher College in Rochester, NY teaches a course called *Mathematical Explorations in the Sciences* that engages students in inquiry based learning in mathematical modeling, and the discovery of fundamental mathematical relationships in scientific contexts. This is a mathematics course where the basic concepts of both physics and chemistry are studied conceptually and mathematically. In one activity, students explore how quadratic functions behave by collecting data on free falling objects and perform curve-fitting regressions using TI-83 graphing calculators. Students prepare reports on their findings the same as a lab report in science classes, using TI software and Geometer Sketchpad (Nrayan, 1991).

The movement toward teaching mathematics by inquiry is inspired by the measured improvement in conceptual understanding in science education, and inability of students to apply math skill to other subject areas. Even students who excel in math courses have difficulty transferring knowledge to the application in science and other classes, maintaining those courses entirely separate from one another (McDermott, 1974; Tall, 1992, Ciu.200)

Most students taking introductory physics courses, whether it by in high school or college, have little experience working hands-on to manipulate a system of objects or using instruments for taking measurements. Furthermore, they have little experience making concrete observations about everyday phenomena to comprehend mathematical representation of them (Laws, 1991). In the same way that physics instructors are engaging students in hands-on activities and observations, mathematics instructors may enrich their lessons by engaging their students in real life observation and interaction with these applications.

DEVELOPING THE CONCEPT OF THE LIMIT THROUGH VELOCITY AND ACCELERATION

The purpose of this paper is to develop one interdisciplinary activity that emphasizes the strong connection of calculus to physics, through an inquiry based learning experience. Research shows that many students taking calculus for the first time have limited mental images of functions, and difficulties in translating real-world problems into calculus formulation. (Tall, 1992) This lesson will demonstrate the specific use of calculus in analyzing the motion of an object in a typical introductory physics kinematics lesson and attempt to anchor abstract calculus concepts of the derivative and the limit, to a physical observable context.

In this activity, students will analyze the three components of motion; position, velocity and acceleration of an object undergoing constant acceleration.

This project is intended for students who have completed, or nearly completed an introductory Calculus I course, as they will be expected to calculate derivatives as well as present their calculations using appropriate mathematical language and notation.

The project will be a discovery based learning experience where students will work in teams and present their findings to the larger group.

Students will keep a journal and be required to provide written descriptions of their observations.

Students will derive the mathematical formulas by analyzing data collected on a free falling object.

Students will employ the use of calculus in their analysis to obtain an accurate model describing the motion of a free falling object.

Students will be required to present the detailed mathematical derivation of these calculations.

Students will pursue the limit of an infinitely small time interval in measuring the instantaneous velocity of their object in motion.

ANALYZING VERTICAL ACCELERTED MOTION OF AN OBJECT IN FREE-FALL

How can we measure the precise position and velocity and acceleration of an accelerating object?

What measurements do you need to take to solve this problem?

These questions are posed to the class to lead them to brainstorm some method of marking the position of the object at various moments in time. Students are placed in groups of four. Each group will work through 3 stages of the activity. Each stage leads to more precise measurement of velocity, demonstrating the idea of taking the limit of a slope function as the time interval approaches zero. Groups will gather raw data on the position of an object undergoing constant acceleration in free fall.

The first stage of the activity is meant to put students through a rigor of manual data collection using stopwatches and tape measure, and calculating the slope of a line between two points by hand (APA 6). The use of technology is intentionally limited in this stage in order to foster a connection between the students and the measurements and help students develop a sense of scale. The act of observing the ball fall and taking measurements through each trial anchors a physical image for students analyzing data relating to these intervals. Data collected in this stage is crude as students record clock readings along the path of a ball being dropped from various heights. This is a standard lesson in most introductory physics courses, demonstrating the effect of gravity.

This lesson can be done in the classroom or on the athletic field bleachers or an open stairwell or a second story balcony. Students stretch a tape measure vertically along the path of the ball, and record time measurements using stopwatches, dropping the ball from increasing heights to simulate *stop-motion*.

Mathematical analysis relies heavily upon interpreting graphs. Sufficient time should be spent establishing students' ability to read and to create meaningful graphical representations. The TI-83+ graphing calculator, or newer model is standard instrument in today's math classroom and students should be well practiced in plotting graphs and performing a curve-fitting regression from a set of given data. (NYS Staandards A2.R.1)

Students begin by dropping the ball from a height of one foot above the floor and measure the time it takes for it to land; then from a height of 8ft, then 13 ft, 18ft, 23ft, and so on. Clock readings on short falls to be extremely rough estimates due to human reaction time. Students create a data table like the one shown.

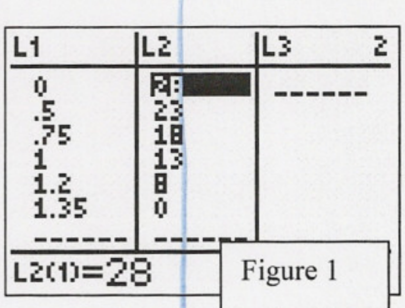


Figure 1

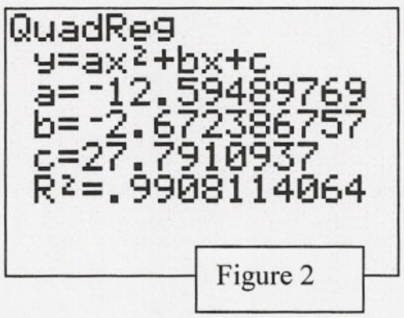


Figure 2

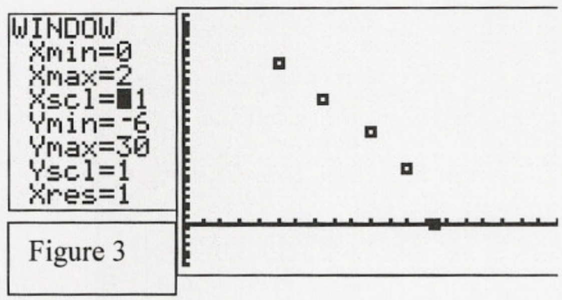


Figure 3

Above is a sample of data collected by a group of students.

Figure 1 shows raw data entered into columns, L1 and L2, where L1 represents clock readings after the ball was released from rest; L2 represents the position of the ball, where the ground level is designated as position zero (0).

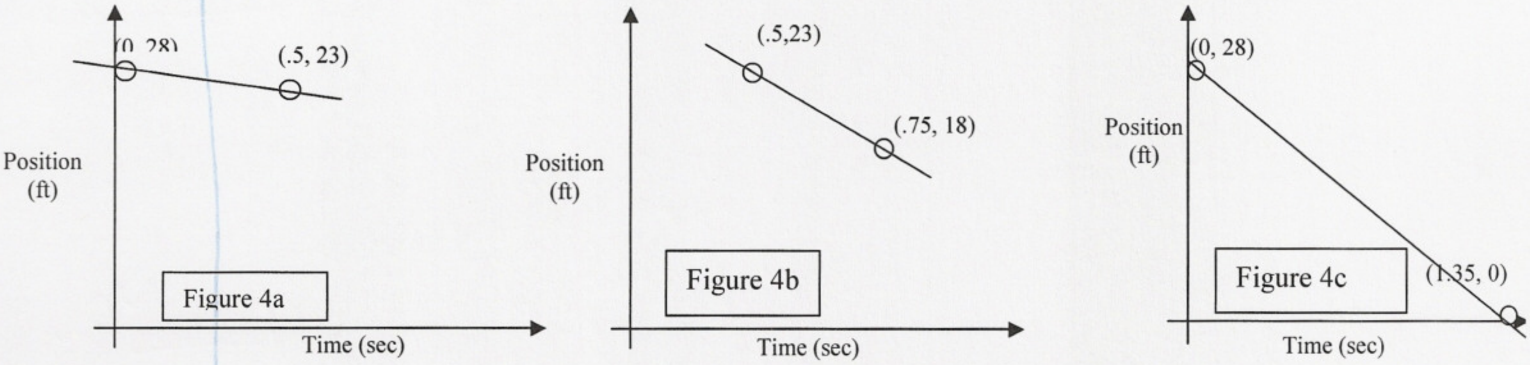
Figure 2 shows the Quadratic Regression equation derived by the machine, including the R^2 value representing the accuracy of the equation to modeling the data.

Figure 3 shows a graph of the data, where time (sec) is plotted along the horizontal axis and position (ft) plotted along the vertical axis.

CALCULATING AVERAGE VELOCITY

Students examine points along the parabolic curve of their *position vs. time* graph in Figure 3 to determine that the slope of the line connecting any two points is equal to the average rate of change in the object's position; or in other words, the average velocity of the object between those two points. A line is drawn between each set of data points, as shown below, and students compile a table of average velocities along the ball's path. Each rate is calculated using the familiar slope formula,

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta position}{\Delta time} = average \quad velocity = \frac{\Delta d}{\Delta t},$$



Slopes should be calculated along large intervals such as (0.28) and (1.35,0), as well as between each two consecutive data points. Some toil should be spent calculating these rates without the aid of the Calculus. (APC.6)

Students are then asked to discuss what is meant by "average velocity." They are lead toward defining *average velocity* as, the uniform velocity an object would have undergone, equivalent to the same position change during the same time interval (Arons, 1997), and asked to articulate written explanations on the values and meaning of each of these slopes in comparison to one another. (APC.1)

Stage 2:

"How can we more accurately measure the velocity of the accelerating ball along its path?"

Students are then challenged to calculate a more accurate measurement of the ball's velocity along its path. Keeping true to a scientific approach, this stage seeks to eliminate one variable; that of error created by human reaction time in measurement. Groups will then repeat the data gathering process; this time using a motion sensor linked to *Logger Pro* or *TI-Interactive* software to track the motion of a falling object.

Here, the motion sensor is used merely to increase the accuracy of measured values from Stage 1. This stage of the activity is intended to familiarize students with using the motion sensor and affirm that the device is taking the same measurements as students took earlier themselves. The instructor should take this opportunity to remind students how the electronic device works by marking an object's position every *x*, number of seconds. At this point, students taking introductory physics should already be familiar with working with motion sensors from previous activities, though students may have considered the device in a mathematics context.

The device should be programmed to record position every .20 seconds (approximately intervals in stage 1). The motion sensor should be positioned on the ground directly beneath the point of release to record the motion of the ball falling vertically until it hits the ground. When linked to a computer running *Logger Pro* or *TI –Interactive* software, data will be collected and displayed along with the *position vs. time graph*. This electronically produced graph is identical to the graphs students produced by hand in stages 1, except that it contains more accurate readings.

L1	L2	L3 2
0	28	------
.2	27.3	
.4	25.4	
.6	22.2	
.8	17.8	
1	12	
1.2	5	

L2(1)=28

Figure 5

```
QuadReg
 y=ax²+bx+c
 a=-15.89285714
 b=-.0714285714
 c=27.97857143
 R²=.9999903556
```

Figure 6

```
WINDOW
 Xmin=0
 Xmax=2
 Xscl=.1
 Ymin=-6
 Ymax=30
 Yscl=1
 Xres=1
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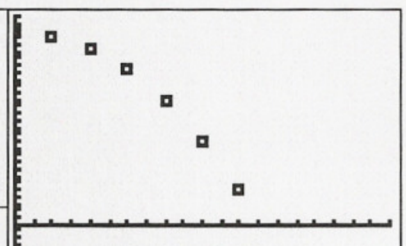

Figure 7

Above is a sample of data generated by using a motion sensor tracking the object falling from a height of 28 ft. Notice the calculated quadratic function more closely matches the accepted function for the position of an object in free-fall.

Figure 5 shows raw data entered into columns, L1 and L2, where L1 represents clock readings after the ball was released from rest; L2 represents the position of the ball, where the ground level is designated as position zero (0).
Figure 6 shows the Quadratic Regression equation derived by the machine, including the R^2 value representing the accuracy of the equation to modeling the data.
Figure 7 shows a graph of the data, where time (sec) is plotted along the horizontal axis and position (ft) plotted along the vertical axis.

It should be recognized that velocities calculated is stage 2 by using the slope formula between data points are still average velocities between two points in time and do not accurately represent the velocity of the object at each moment during that time interval, although an improvement from those calculated in Stage 1. Groups will then conclude that to obtain an even more precise measure, additional data points should be taken.

Students create a new position vs. time graph and repeat their analysis. Students are then challenged to calculate slopes using the function derived by the TI software.
The velocity of the object in free fall during the time interval, $[t,\ t+.20]$ can be calculated by,

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{\left(-15.9(t+.20)^2 + 27.9\right) - \left(-15.9(t)^2 + 27.9\right)}{(t+.20) - (t)}$$

Stage 3

Keeping in mind at all times, that the ball's velocity is constantly increasing we conclude that increasing the number of measurements and improving their accuracy will yield a better approximation of the ball's velocity along its path. Students should think of this in terms to decreasing time intervals more so than increasing the number of data points. (Arons, 1997)

The procedure is then repeated in Stage 3, where students record clock readings on smaller time intervals using a motion sensor. The device should first be set to take readings every 0.20 sec ', matching approximately the interval easured by hand in Stage 1, then In successive trials, the time intervals are gradually reduced until eventually the device will read every .001 seconds. This process leads students from the finite number of data points, toward an ample, approaching infinite, number of data points for analysis.

Using the software, students can zoom in on a section of the graph and calculate the slope of a line connecting two consecutive data points just as in stages 1 and 2. Students should notice that reducing the time intervals leads to a more accurate picture of our scenario by finding more frequent average velocity values of the falling object, and these ideas should be tied to a discussion on the use of calculus in describing continuous motion.

CALCULATING INSTATANEOUS VELOCITY

In their stopwatch trails, students gather a finite number of data points and evaluate average velocities between each. In repeating the process a second time using motion sensors, students make the same observations but more accurately, and on shorter time intervals, allowing them are more accurate description of the object's motion.

The procedure of decreasing the duration of time intervals is intended to help develop the concept of the limit in calculus as Δt approaches zero. A common difficulty among calculus students involves the concept of quantities becoming infinitesimally small, or whether the limit can actually be reached (Tall, 1992). One of the difficulties students have in the transitioning from algebra to calculus is in moving from concrete values to abstract notions of values in calculus. By insistently defining velocity over a certain interval we begin to break the students' misconception about how velocity is commonly defined. As students reduce the duration between measurements, we give tangible meaning to operation of Δt approaching zero. When students are challenged even further still, they themselves conceive the notion of the limit of the function as Δt approaching zero.

Decreasing the duration of time intervals further leads to the formulation of the Fundamental Theorem of Calculus for Derivatives, (APC.2, APC.5)

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}, \text{ and if } t_2 = t + \Delta t, \text{ then, } \frac{f(t + \Delta t) - f(t)}{(t + \Delta t) - t}.$$

And finally, instantaneous velocity can be measured by, $\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$.

ANALYZING HORIZONTAL ACCELERTED MOTION OF AN OBECT ATTACHED TO A SPRING

The lesson should involve multiple scenarios illustrating the same concepts. For this purpose, students will perform analysis on both a vertical and horizontal scenario of accelerated motion. The first activity has students analyze the motion of a free-falling object in one-dimensional vertical motion, where students will eventually derive the acceleration of gravity. The second activity has students repeat their analysis on a horizontal surface where a block is accelerated by the recoil of a stretched spring. The purpose of this follow up exercise is to demonstrate the concept of acceleration in multiple contexts to help students in applying classroom knowledge outside the classroom.

CONCLUSION

This project is designed to intertwine the language of mathematics with the concepts of physics and the observable phenomena of an object in motion. Our aim is to strengthen students' conceptual understanding of position, velocity and acceleration; how they relate to each other and the distinctions between each by taking advantage of the inherent overlapping cross-curricular opportunities of calculus and physics to demonstrate the complex mathematical concepts calculus and foster an appreciation for the role of calculus in the study of physics.

David Tall, of The Mathematics Education Research Center in Quebec writes that students in his study who were engaged in active learning earned the "*same scores as control students in traditional skills, but significantly higher on a questions requiring* *conceptual understanding"* (*Tall, 1992*). He goes on to advocate, an effective method of improving students' comprehension of abstract concepts is through *active,* rather than *passive learning* in an *experience-discovery approach*, as students have d*ifficulty translating real-world problems into calculus formulation.* (*Tall, 1992*).

Several key stumbling points of calculus students are addressed in this procedure. The stages and gradual progression of this project helps lead students to connecting the bridge between algebra and calculus. The calculations begin as simple finite arithmetic, to eventually make the leap to a concept of infinitesimal time span. S*tudents who often have difficulty over whether the limit can actually be reached, and the passage from finite to infinite, in understanding "what happens at infinity"* can certainly relate with the concept of continuous time. (*Tall, 1992*)

Such subjects present difficulties to students in not only its language and mechanical operations, but in the ideas themselves. Subjects of a challenging conceptual nature may require many different examples and explanations that demonstrate the concepts from various standpoints.

Universities throughout the country are currently recognizing the positive results of implementing an interactive engagement component to their calculus courses. The Washington University *Calculus Reform Effort* (Basson,. Krantz, &Thornton, 2006) has developed course offered to second semester Calculus students, which combines traditional classroom lecture instruction with an accompanying math lab. The program does not seek a complete departure from the lecture aspect of instruction, but uses the calculus lab to supplement the course. In this lab, students collect data, enter the data to the computer and perform statistical and mathematical analysis to draw conclusions. The Washington University course comes in response to the complaint by instructors from various disciplines that students who perform well in calculus course are not able to apply those skills on the context of their courses (Basson,. Krantz, &Thornton, 2006, p.335) The study finds that classroom lecture and exercise remains as a fundamental element of math instruction. The classroom instructor is free to implement the lab applications to the lecture as they see fit making direct connections between topics and applications. Students' reaction to the course has been overwhelmingly positive as students acknowledge the practical and conceptual benefits of the lab portion of the course.

One of their first calculus labs of the course establishes fitting data to a curve (NYS Staandards A2.R.1) using the *Logger-Pro* equipment and software. In this activity, students create using *position vs. time* graphs and *velocity vs. time* graphs by walking forward and backward in front of a motion sensor, recording measurements through the *Logger-Pro* software. This is an activity used in many physics courses practicing inquiry, described by Arons and Laws among others. Another lab demonstrates the mathematical behavior of an exponential decay function by measuring the temperature of a cup of hot water as it cools, demonstrating Newton's Law of Cooling. This too is a standard topic in both calculus and Intro to Physics courses.

In another example, Dickson College, Carlisle, PA, has developed a two-semester course called *Workshop Calculus* that serves as a substitute for the traditional one semester course of Calculus 1. The course targets students who may have difficulty gasping the concepts of Calculus 1 in a traditional lecture setting. The course is taught through the method of interactive engagement where students are involved in hands-on learning and journaling. Response and results of the course have been extremely positive. (Hastings, 2008)

Education research finds that conducting interactive engagement activities in math classes enhances students' conceptual understanding of mathematics, retention, and the ability to apply their knowledge of math in other settings. Many institutions are taking the initiative to incorporate science-based activities into calculus courses to enhance conceptual understanding and applicability of mathematics. The use of Interactive Engagement increases the effectiveness of conceptually difficult courses. "*I see no reason to doubt that enhanced understanding and retention would result from greater use of IE methods in other science and even non science areas.*" (Hake, 2007. p. 2)

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NYS Board of Education Standards

AP Calculus

http://teacherweb.com/VA/HeritageHS/MrsLKBrooks/h6.stm

APC.1 The student will define and apply the properties of elementary functions, including algebraic… and graph these functions, using a graphing calculator. Properties of functions will include dom[illegible] ranges… zeros, upper and lower bou[illegible] and intervals where the function is increas[illegible]g or decreasing.

13

APC.2 The student will define and apply the properties of limits of functions. Limits will be evaluated graphically and algebraically.

APC.5 The student will investigate derivatives presented in graphic, numerical, and analytic contexts…. The derivative will be defined as the limit of the difference quotient and interpreted as an instantaneous rate change.

APC.6 The student will investigate the derivative at a point on a curve. This will include
a) finding the slope of a curve at a point;
b) using local linear approximation to find the slope of a tangent line to a curve at the point;
c) defining instantaneous rate of change as the limit of average rate of change; and
d) approximating rate of change from graphs and tables of values.

APC.7 The student will analyze the derivative of a function as a function in itself. This will include
a) comparing corresponding characteristics of the graphs of f, f ', and f ";
b) defining the relationship between the increasing and decreasing behavior of f and the sign of f
c) translating verbal descriptions into equations involving derivatives and vice versa;
d) analyzing the geometric consequences of the Mean Value Theorem;
e) defining the relationship between the concavity of f and the sign of f ";
Discussed as the acceleration function.

APC.8 The student will apply the derivative to solve problems.
This will include
c) modeling of rates of change
e) interpretation of the derivative as a rate of change in applied contexts, including velocity, speed, and acceleration;

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BIOGRAPHY — center

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Peter D. Murray was born in Brooklyn, NY. He received his Associates degree in Liberal Arts at Oakton Community College in Chicago in 1996, and Bachelor of Arts degree in Physics at Indiana University of Pennsylvania in 1999. This is his first Master's degree. Peter currently lives in Saratoga Springs, NY where he teaches Calculus and Trigonometry at the Thomas Edison School of Math Science& Technology, The Plaza, Schenectady NY 12308. He may be contacted at <murrayp@schenectady.k12.ny.us>.